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RESEARCH

A study of elliptic gamma function and allies

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Dedicated to Don Zagier, in admiration of his insights on modular, elliptic and polylogarithmic functions.

Abstract

We study analytic and arithmetic properties of the elliptic gamma function

$$\prod_{m,n=0}^{\infty} \frac{1-x^{-1}q^{m+1}p^{n+1}}{1-xq^mp^n}, \qquad |q|,|p|<1,$$

in the regime $p=q$, in particular, its connection with the elliptic dilogarithm and a formula of S. Bloch. We further extend the results to more general products by linking them to non-holomorphic Eisenstein series and, via some formulae of D. Zagier, to elliptic polylogarithms.

Keywords: Theta function, Elliptic gamma function, Elliptic dilogarithm, Elliptic polylogarithm

1 Introduction

For complex z and τ with $\operatorname{Im}\tau > 0$, set $x = e^{2\pi iz}$ and $q = e^{2\pi i\tau}$. Transformation properties of the so-called *short* theta function

$$\theta_0(z;\tau) := \prod_{m=0}^{\infty}(1-x^{-1}q^{m+1})(1-xq^m)$$

under the action of the modular group are well understood. In view of its transparent invariance under translation $\tau \mapsto \tau+1$, the main source of the modular action originates from the τ-involution

$$z \mapsto \hat z = \frac{z}{\tau}, \quad \tau \mapsto \hat\tau = -\frac{1}{\tau}. \tag{1}$$

The related classical transformation of $\theta_0(z;\tau)$ can be recorded as

$$q^{1/12}x^{-1/2}\theta_0(z;\tau) = ie^{-\pi iz\hat z}\hat q^{1/12}\hat x^{-1/2}\theta_0(\hat z;\hat\tau) \tag{2}$$

(see, for example, [3, Section 2]), where we define $\hat x = e^{2\pi i\hat z}$ and $\hat q = e^{2\pi i\hat\tau}$.

Less is known about modular properties of the related product

$$\theta_1(z;\tau) := \prod_{m=0}^{\infty}\frac{(1-x^{-1}q^{m+1})^{m+1}}{(1-xq^m)^m},$$

which naturally comes as the $\sigma = \tau$ specialisation of the elliptic gamma function

$$\Gamma(z;\tau,\sigma) := \prod_{m,n=0}^{\infty} \frac{1 - x^{-1}q^{m+1}p^{n+1}}{1 - xq^m p^n}, \quad \text{where } p = e^{2\pi i\sigma},$$

introduced by Ruijsenaars [5] (see also [3,4]). Namely, we have

$$\theta_1(z;\tau) = \theta_0(z;\tau)\Gamma(z;\tau,\tau) = \Gamma(z+\tau;\tau,\tau).$$

A known functional equation of the elliptic gamma function [3, Theorem 4.1] represents an $\mathrm{SL}_3(\mathbb{Z})$ symmetry of $\Gamma(z;\tau,\sigma)$. The problem of determining its behaviour in the regime $\sigma = \tau$ under $\mathrm{SL}_2(\mathbb{Z})$ transformations is specifically addressed in [2], where the (logarithm of the) infinite product is related to the elliptic dilogarithm via a formula of S. Bloch [1].

Our principal aim in this note is recasting analytic and arithmetic (modular) properties of the function $\theta_1(z;\tau)$ and its relatives, in particular, linking them to non-holomorphic Eisenstein series and the elliptic dilogarithm. This programme is carried out in Sects. 2–4; it gives a new proof of Bloch's formula and related results from [2]. In Sect. 5 we go further to discuss similar features of products that generalise ones for θ_0 and θ_1; their relationship with non-holomorphic Eisenstein series and formulae from [7] allow us to link them to elliptic polylogarithms.

For future record, notice that iterating the transformation $(z,\tau) \mapsto (\hat{z},\hat{\tau})$ twice maps (z,τ) to $(-z,\tau)$ and that

$$\theta_1(-z;\tau) = \frac{1}{\theta_1(z;\tau)} \quad \text{and} \quad \theta_0(-z;\tau) = -x^{-1}\theta_0(z;\tau). \tag{3}$$

2 Period functions

A natural way of measuring failure of weight k modular behaviour under the transformation $(z,\tau) \mapsto (\hat{z},\hat{\tau})$ for a function $f(z,\tau)$ is through the *period* function

$$g(z,\tau) = g_k(z,\tau) := f(\hat{z},\hat{\tau}) - \tau^k f(z,\tau).$$

Lemma 1 *We have*

$$\tau^k g(\hat{z},\hat{\tau}) + (-1)^k g(z,\tau) = \tau^k\big(f(-z,\tau) - (-1)^k f(z,\tau)\big).$$

Observe that the expression in the parentheses on the right-hand side measures the failure of k-parity of $f(z,\tau)$.

Proof We only use $(\hat{\hat{z}},\hat{\hat{\tau}}) = (-z,\tau)$ and $\tau\hat{\tau} = -1$:

$$\begin{aligned}\tau^k g(\hat{z},\hat{\tau}) - g(z,\tau) &= \tau^k\big(f(-z,\tau) - \hat{\tau}^k f(\hat{z},\hat{\tau})\big) + (-1)^k\big(f(\hat{z},\hat{\tau}) - \tau^k f(z,\tau)\big)\\ &= \tau^k\big(f(-z,\tau) - (-1)^k f(z,\tau)\big).\end{aligned}$$

□

The lemma and the parity relation for $\ln\theta_1(z;\tau)$ in (3) imply the following.

Lemma 2 *The function*

$$T(z;\tau) = \tau\ln\theta_1(z;\tau) - \ln\theta_1(\hat{z};\hat{\tau}) \tag{4}$$

satisfies the functional equation

$$T(\hat{z};\hat{\tau}) = \tau^{-1}T(z;\tau).$$

Furthermore, we can relate the function $T(z;\tau)$ to the dilogarithm function

$$\mathrm{Li}_2(x) = -\int_0^x \ln(1-t)\,\frac{\mathrm{d}t}{t}.$$

Lemma 3 *The function* (4) *admits the following representation:*

$$\begin{aligned} T(z;\tau) &= \frac{\pi i(\tau-2z)(1+2\tau z-2z^2)}{12\tau} + z\ln\theta_0(z;\tau) \\ &\quad - \frac{1}{2\pi i}\sum_{m=0}^{\infty}\big(\mathrm{Li}_2(x^{-1}q^{m+1}) - \mathrm{Li}_2(xq^m)\big). \end{aligned}$$

Proof As shown in the proof of Theorem 5.2 in [3],

$$\begin{aligned} \ln\theta_1(z;\tau) &= \ln\theta_0(z;\tau) + \ln\Gamma(z;\tau,\tau) \\ &= -\pi i\lambda(z;\tau) + \ln\frac{\theta_0(z;\tau)}{\theta_0(\hat z;\hat\tau)} \\ &\quad + (\hat\tau-\hat z)\sum_{k=1}^{\infty}\frac{(\hat x^{-1}\hat q)^k}{k(1-\hat q^k)} - \hat z\sum_{k=1}^{\infty}\frac{\hat x^k}{k(1-\hat q^k)} \\ &\quad + \frac{1}{2\pi i}\sum_{k=1}^{\infty}\frac{\hat x^k-(\hat x^{-1}\hat q)^k}{k^2(1-\hat q^k)} - \hat\tau\sum_{k=1}^{\infty}\frac{\hat q^k(\hat x^k-(\hat x^{-1}\hat q)^k)}{k(1-\hat q^k)^2}, \end{aligned}$$

where

$$\lambda(z;\tau) = \frac{z^3}{3\tau^2} - \frac{2\tau-1}{2\tau^2}z^2 + \frac{(\tau-1)(5\tau-1)}{6\tau^2}z - \frac{(\tau-2)(2\tau-1)}{12\tau}$$

and the assumptions $|\hat x|, |\hat x^{-1}\hat q| < 1$ are made to ensure convergence. (The latter can be dropped in the final result by appealing to the analytic continuation in z.) Recalling the transformation (2), using

$$\frac{1}{1-\hat q^k} = \sum_{m=0}^{\infty}\hat q^{mk} \quad\text{and}\quad \frac{\hat q^k}{(1-\hat q^k)^2} = \sum_{m=0}^{\infty} m\hat q^{mk},$$

interchanging summation and summing over k, we obtain

$$\begin{aligned} \ln\theta_1(z;\tau) &= -\pi i\left(\lambda(z;\tau) - \frac12 + \frac{z^2}{\tau} + \frac{\tau}{6} - z + \frac{1}{6\tau} + \frac{z}{\tau}\right) \\ &\quad + \hat z\sum_{m=0}^{\infty}\left(\ln\left(1-\hat x^{-1}\hat q^{m+1}\right) + \ln\left(1-\hat x\hat q^m\right)\right) \\ &\quad - \hat\tau\sum_{m=0}^{\infty}\left((m+1)\ln\left(1-\hat x^{-1}\hat q^{m+1}\right) - m\ln\left(1-\hat x\hat q^m\right)\right) \\ &\quad - \frac{1}{2\pi i}\sum_{m=0}^{\infty}\left(\mathrm{Li}_2\left(\hat x^{-1}\hat q^{m+1}\right) - \mathrm{Li}_2\left(\hat x\hat q^m\right)\right) \\ &= \frac{\pi i}{12}\left((1+2z) - \frac{2z(1+z)(1+2z)}{\tau^2}\right) + \hat z\ln\theta_0\left(\hat z;\hat\tau\right) - \hat\tau\ln\theta_1\left(\hat z;\hat\tau\right) \\ &\quad - \frac{1}{2\pi i}\sum_{m=0}^{\infty}\left(\mathrm{Li}_2\left(\hat x^{-1}\hat q^{m+1}\right) - \mathrm{Li}_2\left(\hat x\hat q^m\right)\right). \end{aligned}$$

(This formula can be alternatively derived from logarithmically differentiating identity (2) with respect to τ and further integrating the result with respect to z.) Substituting $(z/\tau, -1/\tau)$ for (z, τ) translates the result into

$$\begin{aligned}\tau \ln\theta_1(z;\tau) - \ln\theta_1(\hat{z};\hat{\tau}) &= \frac{\pi i(\tau-2z)(1+2\tau z-2z^2)}{12\tau} + z\ln\theta_0(z;\tau)\\ &\quad - \frac{1}{2\pi i}\sum_{m=0}^{\infty}\big(\mathrm{Li}_2\left(x^{-1}q^{m+1}\right) - \mathrm{Li}_2(xq^m)\big),\end{aligned}$$

the desired relation. □

3 Non-holomorphic modularity

Denote

$$A = A(z,\tau) := \frac{z-\overline{z}}{\tau-\overline{\tau}} \in \mathbb{R},$$

so that

$$\hat{A} = A(\hat{z},\hat{\tau}) := \frac{z\overline{\tau}-\overline{z}\tau}{\tau-\overline{\tau}} \in \mathbb{R}$$

and $z = A\tau - \hat{A}$. Define

$$Q(z;\tau) := q^{B_3(A)/3}\prod_{m=0}^{\infty}\frac{(1-xq^m)^{m+A}}{\left(1-x^{-1}q^{m+1}\right)^{m+1-A}} = \frac{q^{B_3(A)/3}\theta_0(z;\tau)^A}{\theta_1(z;\tau)}, \tag{5}$$

where $B_3(t) := t^3 - \frac{3}{2}t^2 + \frac{1}{2}t$ is the third Bernoulli polynomial, $B_3(1-t) = -B_3(t)$, and

$$F_+(z;\tau) := \ln Q(\hat{z};\hat{\tau}) - \tau\ln Q(z;\tau), \quad F_-(z;\tau) := \ln\overline{Q(\hat{z};\hat{\tau})} - \tau\ln\overline{Q(z;\tau)}.$$

It follows then from Lemma 1 and the parity relations (3) that

$$\begin{aligned}\tau F_+(\hat{z};\hat{\tau}) - F_+(z;\tau) &= \tau\big(\ln Q(-z;\tau) + \ln Q(z;\tau)\big)\\ &= \frac{2\pi i}{3}(B_3(-A)+B_3(A))\tau^2 + 2\pi iAz\tau - \pi iA\tau\\ &= -\pi iA\big(2(A\tau - z)+1\big)\tau = -\pi iA(2\hat{A}+1)\tau\end{aligned}$$

and

$$\begin{aligned}\tau F_-(\hat{z};\hat{\tau}) - F_-(z;\tau) &= \tau\big(\ln\overline{Q(-z;\tau)} + \ln\overline{Q(z;\tau)}\big)\\ &= -\frac{2\pi i}{3}(B_3(-A)+B_3(A))\tau\overline{\tau} - 2\pi iA\overline{z}\tau + \pi iA\tau\\ &= \pi iA\big(2(A\overline{\tau} - \overline{z})+1\big)\tau = \pi iA(2\hat{A}+1)\tau.\end{aligned}$$

We summarise our finding in the following claim.

Lemma 4 *We have*

$$\begin{aligned}\tau F_+(\hat{z};\hat{\tau}) - F_+(z;\tau) &= -\pi iA\left(2\hat{A}+1\right)\tau,\\ \tau F_-(\hat{z};\hat{\tau}) - F_-(z;\tau) &= \quad \pi iA\left(2\hat{A}+1\right)\tau.\end{aligned}$$

Lemma 3 leads to the following expansions of the functions F_+ and F_-.

Theorem 1 *We have*

$$F_+(z;\tau) = S(z,\tau) - \frac{1}{2\pi i} L(z,\tau),$$
$$F_-(z;\tau) = -\frac{2\pi i\overline{\tau}(\tau-\overline{\tau})}{3} B_3(A) + \overline{S(z,\tau)} + \frac{1}{\pi}\overline{U(z,\tau)} + \frac{1}{2\pi i}\overline{L(z,\tau)},$$

where

$$L(z,\tau) := \sum_{m=0}^{\infty} \left(\mathrm{Li}_2\left(x^{-1}q^{m+1}\right) - \mathrm{Li}_2(xq^m)\right),$$
$$U(z,\tau) := \sum_{m=0}^{\infty} \left(\ln|x^{-1}q^{m+1}|\ \mathrm{Li}_1\left(x^{-1}q^{m+1}\right) - \ln|xq^m|\ \mathrm{Li}_1(xq^m)\right),$$
$$S(z,\tau) := \frac{-\pi i}{12}(2A-1)\left(6z^2 - 12A\tau z + 6z + 8A^2\tau^2 - 2A\tau^2 - 6A\tau + 1\right).$$

Proof For F_+ substitute the expression of $T(z;\tau)$ from Lemma 3 into the computation

$$\begin{aligned} F_+(z;\tau) &= \ln Q(\hat{z};\hat{\tau}) - \tau \ln Q(z;\tau) \\ &= \frac{2\pi i}{3}\left(B_3(\hat{A})\hat{\tau} - B_3(A)\tau^2\right) + \hat{A}\ln\theta_0(\hat{z};\hat{\tau}) - \left(\hat{A}+z\right)\ln\theta_0(z;\tau) \\ &\quad + \tau\ln\theta_1(z;\tau) - \ln\theta_1(\hat{z};\hat{\tau}). \end{aligned}$$

This leads to the formula

$$F_+(z;\tau) = S(z,\tau) - \frac{1}{2\pi i} L(z,\tau)$$

with

$$\begin{aligned} S(z,\tau) &= \frac{2\pi i}{3}\left(B_3(\hat{A})\hat{\tau} - B_3(A)\tau^2\right) + \hat{A}\pi i\left(\frac{\tau}{6} - \frac{\hat{\tau}}{6} + z\hat{z} - \frac{1}{2} - z + \hat{z}\right) \\ &\quad + \frac{\pi i}{12\tau}(\tau - 2z)(1 + 2\tau z - 2z^2), \end{aligned}$$

and the latter simplifies to the expression given in the statement of Theorem 1 by elementary manipulation.

For F_- we proceed as follows. We have

$$\ln Q(z;\tau) = \frac{2\pi i\tau B_3(A)}{3} - \sum_{m=0}^{\infty}\left((m+1-A)\,\mathrm{Li}_1(x^{-1}q^{m+1}) - (m+A)\,\mathrm{Li}_1(xq^m)\right).$$

Multiply this expression by $\tau - \overline{\tau} = 2i\,\mathrm{Im}\,\tau$ and use $A(\tau-\overline{\tau}) = 2i\,\mathrm{Im}\,z$ to get

$$(\tau-\overline{\tau})\ln Q(z;\tau) = \frac{2\pi i\tau(\tau-\overline{\tau})B_3(A)}{3} - \frac{1}{\pi}U(z,\tau).$$

Now, notice

$$\overline{(\tau-\overline{\tau})\ln Q(z;\tau)} = F_-(z;\tau) - \overline{F_+(z;\tau)}$$

to deduce the expression for F_- as in the theorem. □

A consequence of this expansion is the invariance of

$$F(z;\tau) := \frac{F_+(z;\tau) + F_-(z;\tau)}{2} = \ln|Q(\hat{z};\hat{\tau})| - \tau\ln|Q(z;\tau)|$$

under translation $\tau \mapsto \tau + 1$.

Lemma 5 *We have*

$$F_+(z;\tau+1)-F_+(z;\tau)=-\big(F_-(z;\tau+1)-F_-(z;\tau)\big).$$

Proof The functions $L(z,\tau)$ and $U(z,\tau)$ (hence their complex conjugates) are clearly invariant under translation $\tau\mapsto\tau+1$. The result follows from noticing that

$$\begin{aligned}2\,\mathrm{Re}\,S(z,\tau)+\frac{2\pi i\overline{\tau}(\tau-\overline{\tau})}{3}B_3(A)&=\frac{-\pi i(\tau-\overline{\tau})^2A(1-A)(1-2A)}{6}\\&=\frac{-\pi i(\tau-\overline{\tau})^2}{3}B_3(A)\end{aligned}$$

is also invariant under the transformation. □

We summarise the results in this section as follows.

Theorem 2 *The weight* 1 *period function*

$$\begin{aligned}F(z;\tau)&=\ln|Q(\hat{z};\hat{\tau})|-\tau\ln|Q(z;\tau)|\\&=\frac{1}{2\pi}\sum_{m=0}^{\infty}\big(\ln|x^{-1}q^{m+1}|\,\overline{\mathrm{Li}_1\,(x^{-1}q^{m+1})}-\ln|xq^m|\,\overline{\mathrm{Li}_1(xq^m)}\,\big)\\&\quad-\frac{\pi i(\tau-\overline{\tau})^2}{6}B_3(A)-\frac{1}{2\pi i}\,\mathrm{Im}\sum_{m=0}^{\infty}\big(\mathrm{Li}_2(x^{-1}q^{m+1})-\mathrm{Li}_2(xq^m)\big)\end{aligned}$$

of $\ln|Q(z;\tau)|$ *satisfies*

$$\tau F(\hat{z};\hat{\tau})=F(z;\tau)\quad\textit{and}\quad F(z;\tau)=F(z;\tau+1).$$

In other words, it behaves like a Jacobi form of weight 1 *on* $\mathrm{SL}_2(\mathbb{Z})$.

4 Elliptic dilogarithm

Theorem 2 provides a natural link between the period function $F(z;\tau)$ and the elliptic dilogarithm [7]

$$D(q;x):=\sum_{m\in\mathbb{Z}}D(xq^m)=\sum_{m=0}^{\infty}\big(D(xq^m)-D(x^{-1}q^{m+1})\big)$$

together with its companion

$$J(q;x):=\sum_{m=0}^{\infty}\big(J(xq^m)-J(x^{-1}q^{m+1})\big)+\frac{\log^2|q|}{3}B_3\bigg(\frac{\log|x|}{\log|q|}\bigg),$$

where

$$D(x):=\ln|x|\,\arg(1-x)+\mathrm{Im}\,\mathrm{Li}_2(x)=-\ln|x|\,\mathrm{Im}\,\mathrm{Li}_1(x)+\mathrm{Im}\,\mathrm{Li}_2(x)$$

denotes the Bloch–Wigner dilogarithm and

$$J(x):=\ln|x|\,\ln|1-x|=-\ln|x|\,\mathrm{Re}\,\mathrm{Li}_1(x)$$

its companion. Namely, the expansion in the theorem can be stated as

$$F(z;\tau)=\frac{1}{2\pi i}\big(D(q;x)+iJ(q;x)\big). \tag{6}$$

This is essentially the result discussed in [2, Section 1].

Viewing now z as an element of the lattice $\mathbb{R}+\mathbb{R}\tau$, so that A and $\hat{A}$ in the representation $z=-\hat{A}+A\tau$ are fixed, we find out that the τ-derivative

$$\frac{1}{2\pi i}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\ln Q(z;\tau)=q\frac{\mathrm{d}}{\mathrm{d}q}\ln Q(z;\tau)$$

is the Eisenstein series

$$\frac{i}{4\pi^3}\sum_{m,n\in\mathbb{Z}}{}' \frac{e^{2\pi i(m\hat{A}+nA)}}{(m\tau+n)^3}$$

of weight 3, where the notation $\sum'$ indicates omitting the term $m=n=0$ from the summation. Integrating we obtain

$$\ln Q(z;\tau)=\frac{1}{4\pi^2}\sum_{m,n\in\mathbb{Z}}{}' \frac{e^{2\pi i(m\hat{A}+nA)}}{m(m\tau+n)^2}$$

implying

$$\begin{aligned}\ln|Q(z;\tau)|&=\frac{1}{2}\big(\ln Q(z;\tau)+\ln\overline{Q(z;\tau)}\big)\\&=\frac{1}{8\pi^2}\sum_{m,n\in\mathbb{Z}}{}' e^{2\pi i(m\hat{A}+nA)}\bigg(\frac{1}{m(m\tau+n)^2}-\frac{1}{m(m\overline{\tau}+n)^2}\bigg)\\&=\frac{1}{2\pi^2}\sum_{m,n\in\mathbb{Z}}{}' e^{2\pi i(m\hat{A}+nA)}\,\frac{i\,m\operatorname{Im}\tau\,(m\operatorname{Re}\tau+n)}{m(m\tau+n)^2(m\overline{\tau}+n)^2}\\&=\frac{i\operatorname{Im}\tau}{2\pi^2}\sum_{m,n\in\mathbb{Z}}{}' \frac{e^{2\pi i(m\hat{A}+nA)}(m\operatorname{Re}\tau+n)}{|m\tau+n|^4}.\end{aligned}$$

This is equation (7) in [2]. Since $\hat{z}=z/\tau=A-\hat{A}/\tau=A+\hat{A}\hat{\tau}$, it follows that

$$\begin{aligned}\ln|Q(\hat{z};\hat{\tau})|&=\frac{i\operatorname{Im}\hat{\tau}}{2\pi^2}\sum_{m,n\in\mathbb{Z}}{}' \frac{e^{2\pi i(-mA+n\hat{A})}(m\operatorname{Re}\hat{\tau}+n)}{|m\hat{\tau}+n|^4}\\&=\frac{i\operatorname{Im}\tau}{2\pi^2|\tau|^2}\sum_{m,n\in\mathbb{Z}}{}' \frac{e^{2\pi i(n\hat{A}-mA)}(-m(\operatorname{Re}\tau)/|\tau|^2+n)}{|n-m/\tau|^4}\\&=\frac{i\operatorname{Im}\tau}{2\pi^2}\sum_{m,n\in\mathbb{Z}}{}' \frac{e^{2\pi i(n\hat{A}-mA)}(n|\tau|^2-m\operatorname{Re}\tau)}{|n\tau-m|^4}\\&=\frac{i\operatorname{Im}\tau}{2\pi^2}\sum_{m,n\in\mathbb{Z}}{}' \frac{e^{2\pi i(m\hat{A}+nA)}(m|\tau|^2+n\operatorname{Re}\tau)}{|m\tau+n|^4}\\&=\frac{\operatorname{Im}\tau}{2\pi^2}\sum_{m,n\in\mathbb{Z}}{}' \frac{e^{2\pi i(m\hat{A}+nA)}((m\operatorname{Re}\tau+n)\tau i+(m\tau+n)\operatorname{Im}\tau)}{|m\tau+n|^4}\end{aligned}$$

implying

$$\ln|Q(\hat{z};\hat{\tau})|-\tau\,\ln|Q(z;\tau)|=\frac{(\operatorname{Im}\tau)^2}{2\pi^2}\sum_{m,n\in\mathbb{Z}}{}' \frac{e^{2\pi i(m\hat{A}+nA)}(m\tau+n)}{|m\tau+n|^4}.$$

The latter is a (non-holomorphic) modular form of weight 1, and combined with equation (6) is the formula of Bloch mentioned previously.

Theorem 3 (Bloch's formula [1,2,7]) *For $z = A\tau - \hat{A}$, we have*

$$\begin{aligned} F(z;\tau) &= \frac{1}{2\pi i}\big(D(q;x) + iJ(q;x)\big) \\ &= \frac{(\operatorname{Im}\tau)^2}{2\pi^2} \sum_{m,n\in\mathbb{Z}}{}' \frac{e^{2\pi i(m\hat{A}+nA)}(m\tau+n)}{|m\tau+n|^4}. \end{aligned}$$

5 General weight

A natural generalisation of the product in (5) is

$$Q_k(z;\tau) := q^{B_{k+2}(A)/(k+2)} \prod_{m=0}^{\infty} (1 - xq^m)^{(m+A)^k} (1 - x^{-1}q^{m+1})^{(-1)^k(m+1-A)^k}, \tag{7}$$

where $k = 0, 1, 2, \ldots$ and $B_k(t)$ stands for the kth Bernoulli polynomial. Then $Q_0(z;\tau)$ is an arithmetic normalisation of the short theta function $\theta_0(z;\tau)$ (a Siegel modular unit) and $Q_1(z;\tau)$ coincides with (5). Following the earlier recipe, define

$$\begin{aligned} F_+(z;\tau) = F_{k,+}(z;\tau) &:= \ln Q_k(\hat{z};\hat{\tau}) - \tau^{k-2}\ln Q_k(z;\tau), \\ F_-(z;\tau) = F_{k,-}(z;\tau) &:= \ln \overline{Q_k(\hat{z};\hat{\tau})} - \tau^{k-2}\ln \overline{Q_k(z;\tau)} \end{aligned}$$

and $F_k(z;\tau) := \frac{1}{2}\big(F_{k,+}(z;\tau) + F_{k,-}(z;\tau)\big)$. Then from Lemma 1 we deduce the following generalisation of Lemma 4.

Lemma 6 *We have, for $k \ge 1$,*

$$\begin{aligned} \tau^k F_+(\hat{z};\hat{\tau}) + (-1)^k F_+(z;\tau) &= \quad (-1)^k \pi i A^k (2\hat{A}+1)\tau^k, \\ \tau^k F_-(\hat{z};\hat{\tau}) + (-1)^k F_-(z;\tau) &= -(-1)^k \pi i A^k (2\hat{A}+1)\tau^k. \end{aligned}$$

Proof Apply Lemma 1 and the relation

$$B_{k+2}(-t) - (-1)^k B_{k+2}(t) = (-1)^k (k+2) t^{k+1}.$$ □

We further use that the τ-derivative of $\ln Q_k(z;\tau)$ is an Eisenstein series.

Lemma 7 *For $k \ge 1$,*

$$\ln Q_k(z;\tau) = \frac{(-1)^k k!}{(2\pi i)^{k+1}} \sum_{m,n\in\mathbb{Z}}{}' \frac{e^{2\pi i(m\hat{A}+nA)}}{m(m\tau+n)^{k+1}},$$

where $z = -\hat{A} + A\tau$.

Proof Consider $\tilde{Q}_k(A,\hat{A};\tau) := Q_k(A\tau - \hat{A};\tau)$ as a function of real variables $A, \hat{A}$ and complex variable τ. The τ-derivative

$$G_{k+2}(A,\hat{A};\tau) := \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\tau} \ln Q_k(A,\hat{A};\tau) = q\frac{\mathrm{d}}{\mathrm{d}q} \ln Q_k(A,\hat{A};\tau)$$

is seen to be the Eisenstein series

$$E_{k+2}(A,\hat{A};\tau) := \frac{(-1)^{k+1}(k+1)!}{(2\pi i)^{k+2}} \sum_{m,n\in\mathbb{Z}}{}' \frac{e^{2\pi i(m\hat{A}+nA)}}{(m\tau+n)^{k+2}}$$

of weight $k+2$. This is true for $k=1$ (see Sect. 4), while for $k \geq 1$ we observe the functional equation

$$\frac{\partial}{\partial \hat{A}} E_{k+3}(A, \hat{A}; \tau) = \frac{\partial}{\partial \tau} E_{k+2}(A, \hat{A}; \tau).$$

The equality $G_{k+2}(A, \hat{A}; \tau) = E_{k+2}(A, \hat{A}; \tau)$ then follows by induction on k using the fact that the constant terms of both functions at $\tau = \infty$ (or $q = 0$) agree.

Integrating we obtain

$$\ln Q_k(A, \hat{A}; \tau) = \frac{(-1)^k k!}{(2\pi i)^{k+1}} \sum_{m,n\in\mathbb{Z}}{}' \frac{e^{2\pi i(m\hat{A}+nA)}}{m(m\tau+n)^{k+1}}.$$

Since both sides continuously depend on A and $\hat{A}$, the formula remains valid also for $\ln Q_k(z;\tau)$. $\square$

As in our computation in Sect. 4 we obtain

$$\begin{aligned}
\ln|Q_k(z;\tau)| &= \frac{(-1)^k k!}{2(2\pi i)^{k+1}} \sum_{m,n\in\mathbb{Z}}{}' e^{2\pi i(m\hat{A}+nA)} \left(\frac{1}{m(m\tau+n)^{k+1}} - \frac{1}{m(m\overline{\tau}+n)^{k+1}} \right) \\
&= \frac{(-1)^k k!}{2(2\pi i)^{k+1}} \sum_{m,n\in\mathbb{Z}}{}' \frac{e^{2\pi i(m\hat{A}+nA)}(\overline{\tau}-\tau)}{(m\tau+n)^{k+1}(m\overline{\tau}+n)^{k+1}} \sum_{j=0}^{k} (m\tau+n)^j (m\overline{\tau}+n)^{k-j} \\
&= -\frac{i^k k!\,\mathrm{Im}\,\tau}{(2\pi)^{k+1}} \sum_{j=0}^{k} \sum_{m,n\in\mathbb{Z}}{}' \frac{e^{2\pi i(m\hat{A}+nA)}}{(m\tau+n)^{k-j+1}(m\overline{\tau}+n)^{j+1}}
\end{aligned}$$

and

$$\begin{aligned}
\ln|Q_k(\hat{z};\hat{\tau})| &= -\frac{i^k k!\,\mathrm{Im}\,\tau}{(2\pi)^{k+1}|\tau|^2} \sum_{j=0}^{k} \sum_{m,n\in\mathbb{Z}}{}' \frac{e^{2\pi i(-mA+n\hat{A})}}{(n-m/\tau)^{j+1}(n-m/\overline{\tau})^{k-j+1}} \\
&= -\frac{i^k k!\,\mathrm{Im}\,\tau}{(2\pi)^{k+1}} \sum_{j=0}^{k} \sum_{m,n\in\mathbb{Z}}{}' \frac{e^{2\pi i(m\hat{A}+nA)}\tau^{k-j}\,\overline{\tau}^j}{(m\tau+n)^{k-j+1}(m\overline{\tau}+n)^{j+1}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
F_k(z;\tau) &= \ln|Q_k(\hat{z};\hat{\tau})| - \tau^k \ln|Q_k(z;\tau)| \\
&= \frac{i^k k!\,\mathrm{Im}\,\tau}{(2\pi)^{k+1}} \sum_{j=0}^{k} \tau^{k-j}(\tau^j - \overline{\tau}^j) \sum_{m,n\in\mathbb{Z}}{}' \frac{e^{2\pi i(m\hat{A}+nA)}}{(m\tau+n)^{j+1}(m\overline{\tau}+n)^{k-j+1}} \\
&= \frac{i^k k!}{2(2\pi)^k(\tau-\overline{\tau})^k} \sum_{j=1}^{k} \tau^{k-j}(\tau^j - \overline{\tau}^j) D_{j+1,k-j+1}(q;x) \\
&= \frac{i\,k!}{(4\pi\,\mathrm{Im}\,\tau)^k} \sum_{j=1}^{k} \tau^{k-j}\,\mathrm{Im}(\tau^j)\,D_{j+1,k-j+1}(q;x),
\end{aligned}$$

where

$$D_{a,b}(q;x) := \frac{(\tau-\overline{\tau})^{a+b-1}}{2\pi i} \sum_{m,n\in\mathbb{Z}}{}' \frac{e^{2\pi i(m\hat{A}+nA)}}{(m\tau+n)^a(m\overline{\tau}+n)^b} \tag{8}$$

for positive integers a and b.

Finally, observe that the non-holomorphic Eisenstein series (8) can be identified with the elliptic polylogarithms using a formula of Zagier [7, Proposition 2]. This leads to the following general result.

Theorem 4 *For $k \ge 1$ and $z = A\tau - \hat{A}$, we have*

$$\ln|Q_k(\hat{z};\hat{\tau})| - \tau^k \ln|Q_k(z;\tau)| = \frac{i\,k!}{(4\pi\,\mathrm{Im}\,\tau)^k} \sum_{j=1}^{k} \tau^{k-j}\,\mathrm{Im}(\tau^j)\,D_{j+1,k-j+1}(q;x),$$

where

$$D_{a,b}(q;x) = \sum_{m=0}^{\infty} \big(D_{a,b}(xq^m) + (-1)^{a+b} D_{a,b}(x^{-1}q^{m+1})\big) + \frac{(4\pi\,\mathrm{Im}\,\tau)^{a+b-1}}{(a+b)!} B_{a+b}(A)$$

and

$$\begin{aligned} D_{a,b}(x) &= (-1)^{a-1} \sum_{\ell=a}^{a+b-1} 2^{a+b-\ell-1} \binom{\ell-1}{a-1} \frac{(-\ln|x|)^{a+b-\ell-1}}{(a+b-\ell-1)!}\,\mathrm{Li}_\ell(x) \\ &\quad + (-1)^{b-1} \sum_{\ell=b}^{a+b-1} 2^{a+b-\ell-1} \binom{\ell-1}{b-1} \frac{(-\ln|x|)^{a+b-\ell-1}}{(a+b-\ell-1)!}\,\overline{\mathrm{Li}_\ell(x)}. \end{aligned}$$

6 Conclusion

This final (and very short!) part is devoted to highlighting some directions for further research.

In spite of generalisability of the story in Sects. 2–4 to the function

$$F_k(z;\tau) = \ln|Q_k(\hat{z};\hat{\tau})| - \tau^k \ln|Q_k(z;\tau)|,$$

where $k \ge 1$ and the product $Q_k(z;\tau)$ is defined in (7), the case $k = 1$ remains the only one, which is invariant under translation $\tau \mapsto \tau + 1$. At the same time, Lemma 6 implies the transformation

$$\tau^k F_k(\hat{z},\hat{\tau}) = (-1)^{k-1} F_k(z,\tau) \quad \text{for } k = 1, 2, \dots .$$

This consideration does not exclude, however, a possibility for modified products (7) and related functions F_k to exist such that the latter ones have true modular behaviour for each $k \ge 1$. It sounds to us a nice problem to determine such modular objects.

Several arithmetic problems related to the case $k = 1$ (originating from the elliptic gamma function) are still open. Our personal favourites include connection of (5) with the Mahler measure and mirror symmetry; see, for example, observation in [6].

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Conflict of interest

On behalf of all authors, the corresponding author Wadim Zudilin states that there is no conflict of interest.

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